Isomorphism theorems

Recall that by the way we constructed quotient groups we obtained the following result, which is the first of four "isomorphism theorems". These are all fundamental results, and we'll make a lot of use of them in this class and in any future algebra-related class:

The First Isomorphism Theorem: If $\varphi: G \to H$ is a homomorphism, then $\ker\varphi \trianglelefteq G$ and $G/\ker\varphi \cong \underset{\substack{\parallel \\ (\operatorname{im}\varphi)}}{\varphi(G)}$.

We already proved both of these statements.

Cor: $|G : \ker\varphi| = |\varphi(G)|$.

The Second Isomorphism Theorem: Let G be a group, A and B subgroups and $A \leq N_G(B)$. Then

1.) $AB \leq G$

2.) $B \trianglelefteq AB$

3.) $A \cap B \trianglelefteq A$

4.) $AB/B \cong A/A\cap B$.

Pf: 1.) Since $A \leq N_G(B)$ and $B \trianglelefteq N_G(B)$, from the last section we get $AB \leq N_G(B) \leq G \Rightarrow AB \leq G$.

2.) Since $AB \leq N_G(B)$ and $B \trianglelefteq N_G(B)$, every element of AB normalizes B, so $B \trianglelefteq AB$.

3.) By 2.), we can define AB/B.

Define $\varphi : A \longrightarrow AB/B$

by $\varphi(a) = aB$.

This is a homomorphism since $\varphi(a_1 a_2) = a_1 a_2 B = a_1 B a_2 B$.

$a \in \ker \varphi \iff aB = 1B \iff a \in B$.

Thus, $\ker \varphi = A \cap B$, so $A \cap B \trianglelefteq A$

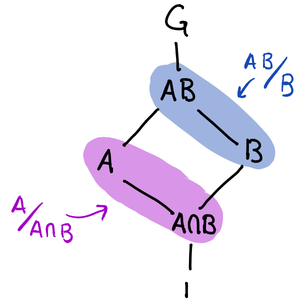
4.) φ is surjective, since for $a \in A$, $b \in B$,

$$(ab)B = aB\underbrace{bB}_{1B} = aB\underbrace{1B}_{\text{identity}} = aB = \varphi(a).$$

Thus, by the first isom. thm, we have

$$A/A\cap B = A/\ker\varphi \cong \varphi(A) = AB/B. \quad \square$$

This is also called the "Diamond Isomorphism Thm", because we have the following picture



The Third Isomorphism Theorem: Let G be a group and H and K normal subgroups of G. Assume $H \leq K$. Then $K/H \trianglelefteq G/H$ and $(G/H)/(K/H) \cong G/K$.

This is also called the "freshman theorem". It is exactly what you get if you thought of these as fractions (but don't think of them as fractions!)

Pf: Define $\varphi: G/H \to G/K$ by $gH \mapsto gK$.

We have to show φ is well-defined. If $gH = g'H$, then $gH \subseteq gK$ and $g'H \subseteq g'K$, so $gH \subseteq gK \cap g'K$, so the intersection is nonempty.

$$\Rightarrow gK = g'K.$$

By construction, φ is a homomorphism, and if $gK \in G/K$, $\varphi(gH) = gK$, so φ is surjective.

$\ker\varphi = \{gH \mid gK = 1K\} = \{gH \mid g \in K\} = K/H$. Thus, $K/H \trianglelefteq G/H$.

By the 1st isom. thm, we get $(G/H)/(K/H) = (G/H)/\ker\varphi \cong \operatorname{im}\varphi = G/K$. $\square$

The final isomorphism theorem describes the relationship between the subgroups of G and the subgroups of G/N. Roughly, it says that the subgroups of G/N correspond to the subgroups of G that contain N. i.e. if you take the lattice of subgroups for G and you collapse everything below N, you get the lattice for G/N. This is also called the "Lattice isomorphism Theorem".

The Fourth Isomorphism Theorem: Let $N \trianglelefteq G$. Then there is a bijection from the set of subgroups A of G which contain N onto the set of subgroups $\overline{A} = A/N$ of G/N. In particular, every subgroup of $\overline{G} = G/N$ is of the form A/N for some $A \leq G$ s.t. $N \leq A$. (Namely, A is the preimage of of A/N in G under the homomorphism $G \to G/N$ sending $g \mapsto gN$.)

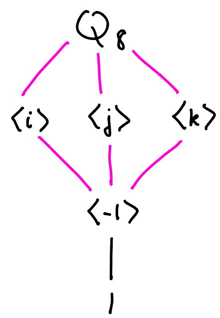
Pf: If $A \leq G$ s.t. $N \leq A$, then G normalizes N so A does as well. Thus $N \trianglelefteq A$. WTS $A/N \leq G/N$.

If $a_1 N, a_2 N \in A/N$, then $a_1 a_2^{-1} \in A$, so $a_1 a_2^{-1} N \in A/N$. Thus $A/N \leq G/N$.

Now suppose $B \leq G/N$. Let $\varphi : G \to G/N$ be the natural projection. Let $B' = \varphi^{-1}(B)$. $B' \leq G$ (the preimage of a subgp is a subgroup) $N \leq B'$ since $N = \varphi^{-1}(1)$, and $B'/N = \varphi(B') = B$. Thus, this is a one-to-one correspondence. $\square$

Ex: Let $G = Q_8$. $\langle -1 \rangle$ is normal since $\langle -1 \rangle = Z(Q_8)$, and every subgroup of the center is always normal.

We draw the lattice below for Q_8, and show the sublattice which corr. to the lattice for $Q_8/\langle -1 \rangle$.



$\mathbb{Z}_2 \times \mathbb{Z}_2$ has 3 proper nontrivial subgps while $\mathbb{Z}/4\mathbb{Z}$ has just 1 (gen. by $\bar{2}$)

Thus, we can see $Q_8/\langle -1\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.